

Homework Assignment II

1. Let $\mathbf{A} = (a_{jk}) \in R^{m \times N}$, $\mathbf{y} \in R^m$, and $\mathbf{x} \in R^N$. Consider ℓ_1 -minimization problem

$$(P_1): \min_{\mathbf{x} \in R^N} \|\mathbf{x}\|_1 \text{ such that } \mathbf{y} = \mathbf{A}\mathbf{x}.$$

Show that there exists at least one solution \mathbf{x}_o of (P1) such that $\|supp(\mathbf{x}_o)\| \leq m$.

2. Let V be a normed vector space with an inner product $\langle \cdot, \cdot \rangle: V \times V \mapsto C$ defined. Prove the famous Cauchy Schwarz inequality of sparse vectors which says for all $\mathbf{v}, \mathbf{w} \in V$,

$$|\langle \mathbf{A}\mathbf{v}, \mathbf{A}\mathbf{w} \rangle| \leq \delta_s \|\mathbf{v}\|_2 \|\mathbf{w}\|_2$$

where $s = |supp(\mathbf{v})| + |supp(\mathbf{w})|$, and $supp(\mathbf{v}) \cap supp(\mathbf{w}) = \emptyset$, and δ_s is the RIP constant of order s for the matrix \mathbf{A} .

3. Recall from lecture that **mutual coherence** for a two orthonormal bases $\mathbf{U} = \{\mathbf{u}_1, \dots, \mathbf{u}_N\}$ and $\mathbf{V} = \{\mathbf{v}_1, \dots, \mathbf{v}_N\}$ is defined as follows

$$\mu(\mathbf{U}, \mathbf{V}) := \max_{1 \leq k, l \leq N} |\langle \mathbf{u}_k, \mathbf{v}_l \rangle|.$$

- (a) Establish that

$$1/\sqrt{N} \leq \mu(\mathbf{U}, \mathbf{V}) \leq 1$$

and these inequalities are sharp (i.e., proving that equality exists).

- (b) For any arbitrary non-zero $\mathbf{b} \in R^N$ such that $\mathbf{b} = \mathbf{U}\boldsymbol{\alpha} = \mathbf{V}\boldsymbol{\beta}$, prove that

$$\|\boldsymbol{\alpha}\|_0 + \|\boldsymbol{\beta}\|_0 \geq 2/\mu(\mathbf{U}, \mathbf{V}).$$

Due date: **February 28** in class