On the Completeness of the Lattice Factorization for Linear-Phase Perfect Reconstruction Filter Banks

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Abstract— **In this letter, we re-examine the completeness of the lattice factorization for M-channel linear-phase perfect reconstruction filter bank (LPPRFB) with filters of the same length $L = KM$ in [1]. We point out that the assertion of completeness in [1] is incorrect. Examples are presented to show that the proposed lattice structure in [1] is *not* complete when $K > 2$. In addition, we verify that the lattice structure in [1] is complete only when $K \leq 2$.**

Index Terms—**Lattice factorization, linear-phase perfect reconstruction filter bank, completeness.**

I. Introduction

Lattice factorization is one of the most attractive methods for the design and implementation of filter banks. An important concept associated with the lattice structure is *completeness*. Completeness of a lattice implies that the lattice can cover all possible solutions for any filter bank possessing certain desired properties, such as perfect reconstruction (PR), paraunitary (PU) and/or linear-phase perfect reconstruction (LPPR). In [1], a lattice structure for an M-channel linear-phase perfect reconstruction filter bank (LPPRFB) with all the filters of the same length $L = KM$ was introduced. When M is even, the proposed lattice structure in [1] was asserted to be complete ([1], Theorem II), i.e., the lattice was supposed to cover all even-channel LPPRFBs with the same filter length. However, in this letter, we point out that the assertion of completeness in [1] is incorrect. Examples are presented to show that the proposed lattice structure in [1] is *not* complete when $K > 2$. In addition, we prove that the lattice structure in [1] is complete only when $K \leq 2$.

Hereafter, we use the following notations to denote certain special matrices. For a positive integer m, $\mathbf{I}_m$, $\mathbf{J}_m$ and $\mathbf{0}_m$ denote the $m \times m$ identity matrix, reversal matrix and null matrix [1], respectively. Moreover, $\mathbf{D}_{2m}$ is a $2m \times 2m$ matrix as follows

$$\mathbf{D}_{2m} = \begin{bmatrix} \mathbf{I}_m & \mathbf{0}_m \\ \mathbf{0}_m & -\mathbf{I}_m \end{bmatrix}. \tag{1}$$

II. Lattice factorization for LPPRFBs

Consider an M-channel (M even, $M = 2m$) LPPRFB with all the analysis and synthesis filters of the same length $L = KM$ each. Let $\mathbf{E}(z)$ and $\mathbf{R}(z)$ denote the corresponding analysis and synthesis polyphase matrix, respectively. $\mathbf{E}(z)$ and $\mathbf{R}(z)$ satisfy the LPPR condition as follows [1]

$$\mathbf{R}(z)\mathbf{E}(z) = z^{-l}\mathbf{I}_M, \qquad l \geq 0, \tag{2}$$

$$\mathbf{E}(z) = z^{-(K-1)}\mathbf{D}_M\mathbf{E}(z^{-1})\mathbf{J}_M, \tag{3}$$

$$\mathbf{R}(z) = z^{-(K-1)}\mathbf{J}_M\mathbf{R}(z^{-1})\mathbf{D}_M. \tag{4}$$

Equation (2) is referred to as the PR property, and equations (3) and (4) are referred to as the LP property. Collectively, (2)-(4) are called the LPPR condition. For this subclass of LPPRFBs, the lattice factorization derived in [1] is as follows:

$$\mathbf{E}(z) = \mathbf{G}_{K-1}(z)\mathbf{G_{K-2}(z)}\cdots\mathbf{G_2(z)G_1(z)E_0}, \tag{5}$$

$$\mathbf{G}_i(z) = \frac{1}{2}\begin{bmatrix} \mathbf{U}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_i \end{bmatrix}\begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & z^{-1}\mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix} \triangleq \frac{1}{2}\Phi_i\mathbf{W}\Lambda(\mathbf{z})\mathbf{W}, \tag{6}$$

and

$$\mathbf{E}_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{U}_0 & \mathbf{U}_0\mathbf{J} \\ \mathbf{V}_0\mathbf{J}_m & -\mathbf{V}_0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{U}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_0 \end{bmatrix}\begin{bmatrix} \mathbf{I} & \mathbf{J} \\ \mathbf{J} & -\mathbf{I} \end{bmatrix}, \tag{7}$$

where $\mathbf{U}_i, \mathbf{V}_i$ (for $i = 0, 1, \ldots, K-1$) are arbitrary $m \times m$ invertible matrices. The synthesis polyphase matrix $\mathbf{R}(z)$ can be obtained by inverting each analysis component one by one in $\mathbf{E}(z)$ [1].

Although the above factorization can structurally enforce both LP and PR properties, it contains redundant free parameters. In [2], a simplified lattice factorization was presented, where all the matrices $\mathbf{V}_i$ (for $i = 1, \cdots, K-1$) can be replaced by the identity matrix $\mathbf{I}_m$ without loss of generality. That is, all $\mathbf{G}_i(z)$ in (6) can be substituted by

$$\mathbf{G}_i(z) = \frac{1}{2}\begin{bmatrix} \mathbf{U}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & z^{-1}\mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix}. \tag{8}$$

The simplified expression in (8) provides a much more efficient way for both design and implementation of LPPRFB. Later, we will use it to analyze the completeness of the lattice factorization.

III. On the completeness of the lattice factorization

A. Review of the original proof

In Theorem II of [1], it was stated that the lattice factorization in (5) is complete, i.e., it covers all possible solutions of even-channel LPPRFBs with filter length of $L = KM$ each. However, in the following, we shall show that this assertion is incorrect. To this end, let us briefly review the original incorrect proof in [1].

The proof of completeness relies on the existence of a building block $\mathbf{G}_{K-1}(z)$ that can reduce the order of $\mathbf{E}(z)$ by 1 at a time while retaining the LPPR property of the reduced-order $\mathbf{F}(z) = \mathbf{G}_{K-1}^{-1}(z)\mathbf{E}(z)$. A critical step in the proof is the *causality* of the factorization, i.e., there always exist invertible matrices $\mathbf{U}_{K-1}$ and $\mathbf{V}_{K-1}$ that can produce a causal $\mathbf{F}(z)$. Let $\mathbf{F}(z) = \sum_{i=0}^{K-2} \mathbf{F}_i z^{-i}$ ($\mathbf{F}_{K-2} \neq 0$) and $\mathbf{E}(z) = \sum_{i=0}^{K-1} \mathbf{E}_i z^{-i}$ ($\mathbf{E}_{K-1} \neq 0$), the proof of causality is equivalent to showing that there always exist two invertible matrices $\mathbf{U}_{K-1}$ and $\mathbf{V}_{K-1}$ such that ((A.4), [1])

$$\begin{bmatrix} \mathbf{U}_{K-1}^{-1} & -\mathbf{V}_{K-1}^{-1} \\ -\mathbf{U}_{K-1}^{-1} & \mathbf{V}_{K-1}^{-1} \end{bmatrix} \cdot \mathbf{E}_0 = \mathbf{0}_M. \tag{9}$$

In [1], it is presumed that (9) is satisfied if $rank(\mathbf{E}_0) \leq M/2$, since, in that case, the dimension of the null space of $\mathbf{E}_0$ is larger than or equal to $M/2$, i.e., $rank(Null(\mathbf{E}_0)) \geq M/2$, in which $Null(\mathbf{E}_0)$ denotes the null space of $\mathbf{E}_0$. It is assumed in [1] that *under this condition, it is possible to choose $M/2$ linearly independent vectors from $\mathbf{E}_0$'s null space to serve as $\begin{bmatrix} \mathbf{U}_{K-1}^{-1} & -\mathbf{V}_{K-1}^{-1} \end{bmatrix}$.*

As previously mentioned, $\mathbf{G}_i(z)$ in (6) can be replaced by $\mathbf{G}_i(z)$ in (8) without loss of generality. Thus, we can rephrase the above assumption in the simplified factorization as follows. If $rank(Null(\mathbf{E}_0)) \geq M/2$, there always exist an invertible matrix $\mathbf{U}_{K-1}$ such that

$$\begin{bmatrix} \mathbf{U}_{K-1}^{-1} & -\mathbf{I}_m \\ -\mathbf{U}_{K-1}^{-1} & \mathbf{I}_m \end{bmatrix} \cdot \mathbf{E}_0 = \mathbf{0}_M. \tag{10}$$

However, such assumption is *not* true.

Now, express $\mathbf{E}_0$ as

$$\mathbf{E}_0 = \begin{bmatrix} \mathbf{E}_u \\ \mathbf{E}_d \end{bmatrix}, \tag{11}$$

where $\mathbf{E}_u$ and $\mathbf{E}_d$ are the upper and lower submatrices of $\mathbf{E}_0$ with size of $M/2 \times M$ each. Substituting (11) into (10) yields

$$\mathbf{E}_u = \mathbf{U}_{K-1}\mathbf{E}_d, \tag{12}$$

which indicates that $\mathbf{E}_u$ can be represented through a linear transform of $\mathbf{E}_d$. In other words, (10) is satisfied *if and only if* the row vectors of $\mathbf{E}_u$ and $\mathbf{E}_d$ span the same space, *the condition $rank(Null(\mathbf{E}_0)) \geq M/2$ alone cannot guarantee the existence of* $\mathbf{U}_{K-1}$ *in* (10). Hence, the original proof of completeness in [1] is incorrect and consequently, the completeness of the lattice factorization needs to be restudied. In the following, through counter examples, we will show that the factorization is *not* complete when $K > 2$. Then, we prove that the factorization is complete when $K \leq 2$.

B. $K > 2$

Actually, (12) is a very strong restriction. When $K > 2$, an LPPRFB can exist even if (12) does not hold. To see this more clearly, we first provide a counter example for $K = 3$. Based on it, other counter examples can be generated for $K > 3$.

B.1 $K = 3$

A counter example for $K = 3$ is as follows. For instance, when $M = 4$, let $\mathbf{E}(z)$ and $\mathbf{R}(z)$ be chosen as

$$\mathbf{E}(z) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & -2 \\ 0 & -2 & 2 & 0 \end{bmatrix} z^{-1} + \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix} z^{-2}, \tag{13}$$

and

$$\mathbf{R}(z) = \begin{bmatrix} -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} \end{bmatrix} + \begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & -\frac{1}{4} & 0 \end{bmatrix} z^{-1} + \begin{bmatrix} -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} \end{bmatrix} z^{-2}. \tag{14}$$

One can verify that $\mathbf{E}(z)$ and $\mathbf{R}(z)$ satisfy

$$\mathbf{R}(z)\mathbf{E}(z) = z^{-2}\mathbf{I}_4, \tag{15}$$

$$\mathbf{E}(z) = z^{-2}\mathbf{D}_4\mathbf{E}(z^{-1})\mathbf{J}_4, \tag{16}$$

$$\mathbf{R}(z) = z^{-2}\mathbf{J}_4\mathbf{R}(z^{-1})\mathbf{D}_4. \tag{17}$$

That is, $\mathbf{E}(z)$ and $\mathbf{R}(z)$ satisfy the LPPR condition with $l = 2$, $K = 3$ and $M = 4$ in (2)-(4). Moreover, from (13), it is easy to see that $rank(\mathbf{E}_0) = 4/2 = 2$. However, in this example, the basis vector of $\mathbf{E}_u$ is $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$, while the basis vector of $\mathbf{E}_d$ is $\begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}$. They do not span the same space. Thus, $\mathbf{U}_2$ does not exist. As a result, $\mathbf{E}(z)$ can not be expressed in terms of (5). This example means that the factorization cannot cover all the solutions when $K = 3$. Based on it, we can also find other counter examples for $K > 3$ as follows.

B.2 $K > 3$

When K is odd ($K = 2K_0 + 1$, $K_0 \geq 1$), let the corresponding analysis bank $\mathbf{E}^o(z) = \sum_{i=0}^{K-1} \mathbf{E}_i^o z^{-i}$ and synthesis bank $\mathbf{R}^o(z) = \sum_{i=0}^{K-1} \mathbf{R}_i^o z^{-i}$ be chosen as

$$\mathbf{E}^o(z) = \mathbf{E}(z^{K_0}) \qquad \text{and} \qquad \mathbf{R}^o(z) = \mathbf{R}(z^{K_0}), \tag{18}$$

in which $\mathbf{E}(z)$ and $\mathbf{R}(z)$ are the same as in (13) and (14), respectively. From (15)-(18), one can derive that

$$\begin{aligned} &\mathbf{R}(z^{K_0})\mathbf{E}(z^{K_0}) = z^{-2K_0}\mathbf{I}_4 \\ \Longrightarrow\ &\mathbf{R}^o(z)\mathbf{E}^o(z) = z^{-2K_0}\mathbf{I}_4, \end{aligned} \tag{19}$$

$$\begin{aligned} &\mathbf{E}(z^{K_0}) = z^{-2K_0}\mathbf{D}_4\mathbf{E}(z^{-K_0})\mathbf{J}_4 \\ \Longrightarrow\ &\mathbf{E}^o(z) = z^{-2K_0}\mathbf{D}_4\mathbf{E}^o(z^{-1})\mathbf{J}_4, \end{aligned} \tag{20}$$

$$\begin{aligned} &\mathbf{R}(z) = z^{-2K_0}\mathbf{J}_4\mathbf{R}(z^{-K_0})\mathbf{D}_4 \\ \Longrightarrow\ &\mathbf{R}^o(z) = z^{-2K_0}\mathbf{J}_4\mathbf{R}^o(z^{-1})\mathbf{J}_4. \end{aligned} \tag{21}$$

In other words, $\mathbf{E}^o(z)$ and $\mathbf{R}^o(z)$ satisfy the LPPR condition in (2)-(4) with $l = 2K_0$, $K = 2K_0 + 1$ and $M = 4$. But since $\mathbf{E}_0^o = \mathbf{E}_0$, for the same reason as explained in the previous example, $\mathbf{E}^o(z)$ can not be represented through (5), either.

On the other hand, when K is even ($K = 2(K_0 + 1)$, $K_0 \geq 1$), let the analysis bank $\mathbf{E}^e(z) = \sum_{i=0}^{K-1} \mathbf{E}_i^e z^{-i}$ and synthesis bank $\mathbf{R}^e(z) = \sum_{i=0}^{K-1} \mathbf{R}_i^e z^{-i}$ be chosen as

$$\begin{aligned} \mathbf{E}^e(z) &= \mathbf{E}^o(z)diag(\mathbf{I}_2,\ z^{-1}\mathbf{I}_2) \\ \mathbf{R}^e(z) &= diag(z^{-1}\mathbf{I}_2,\ \mathbf{I}_2)\mathbf{R}^o(z), \end{aligned} \tag{22}$$

where $\mathbf{E}^o(z)$ and $\mathbf{R}^o(z)$ are defined in (18) with K_0 taking the same value. Since $\mathbf{E}^o(z)$ and $\mathbf{R}^o(z)$ satisfy the PR property in (2), it is easy to verify that $\mathbf{E}^e(z)$ and $\mathbf{R}^e(z)$ also satisfy the PR property as follows.

$$\begin{aligned} \mathbf{R}^e(z)\mathbf{E}^e(z) &= diag(z^{-1}\mathbf{I}_2,\ \mathbf{I}_2)\mathbf{R}^o(z)\mathbf{E}^o(z)diag(\mathbf{I}_2,\ z^{-1}\mathbf{I}_2) \\ &= z^{-(2K_0+1)}\mathbf{I}_4. \end{aligned} \tag{23}$$

Moreover, noticing the fact that

$$diag(\mathbf{I}_2, z^{-1}\mathbf{I}_2) = z^{-1}\mathbf{J}_4 diag(\mathbf{I}_2, z\mathbf{I}_2)\mathbf{J}_4, \tag{24}$$

and using (20), we can derive that $\mathbf{E}^e(z)$ satisfies the LP property as follows

$$\begin{aligned} \mathbf{E}^e(z) &= \mathbf{E}^o(z)diag(\mathbf{I}_2, z^{-1}\mathbf{I}_2) \\ &= z^{-2K_0}\mathbf{D}_4\mathbf{E}^o(z^{-1})\mathbf{J}_4 z^{-1}\mathbf{J}_4 diag(\mathbf{I}_2, z\mathbf{I}_2)\mathbf{J}_4 \\ &= z^{-(2K_0+1)}\mathbf{D}_4\mathbf{E}^e(z^{-1})\mathbf{J}_4 \\ &= z^{-(K-1)}\mathbf{D}_4\mathbf{E}^e(z^{-1})\mathbf{J}_4. \end{aligned} \tag{25}$$

In a similar way, one can verify that $\mathbf{R}^e(z)$ satisfies the LP property also

$$\begin{aligned} \mathbf{R}^e(z) &= z^{-(2K_0+1)}\mathbf{J}_4\mathbf{R}^e(z^{-1})\mathbf{D}_4 \\ &= z^{-(K-1)}\mathbf{J}_4\mathbf{R}^e(z^{-1})\mathbf{D}_4. \end{aligned} \tag{26}$$

Equations (23) and (25)-(26) mean that $\mathbf{E}^e(z)$ and $\mathbf{R}^e(z)$ satisfy the LPPR condition in (2)-(4) with $l = 2K_0 + 1$, $K = 2(K_0 + 1)$ and $M = 4$. However, from (22), one can compute that

$$\mathbf{E}_0^e = \mathbf{E}_0^o \begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 \end{bmatrix} = \mathbf{E}_0 \begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}. \tag{27}$$

Let $\mathbf{E}_u^e$ and $\mathbf{E}_d^e$ denote the first two rows and the last two rows of $\mathbf{E}_0^e$, respectively, i.e.,

$$\mathbf{E}_u^e = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix} \text{ and } \mathbf{E}_d^e = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}. \tag{28}$$

Clearly, the row vectors of $\mathbf{E}_u^e$ and $\mathbf{E}_d^e$ do not span the same space. Therefore, $\mathbf{E}^e(z)$ cannot be represented in terms of (5), either.

From the above counter examples, we can conclude that the lattice factorization in (5) is *not* complete for $K > 2$.

C. $K \leq 2$

Now, let us study the case when $K \leq 2$. It is easy to see that when $K = 1$, the factorization is complete [1]. Next, we will prove that the factorization is also complete when $K = 2$.

Similarly as in [1], we have to show the existence of a $\mathbf{G}_1^{-1}(z)$ such that $\mathbf{F}(z) = \mathbf{G}_1^{-1}(z)\mathbf{E}(z)$ corresponds to an LPPRFB with $K = 1$. Since the preservation of LPPR property and order reduction property have been verified in [1], we need only to prove causality. The proof of completeness for $K = 2$ is accomplished if we can verify the existence of an invertible matrix $\mathbf{U}_1$ satisfying (10), or equivalently, (12). Just as in [1], for simplicity of exposition, denote $\mathbf{E}(z) = \mathbf{E}_0 + \mathbf{E}_1 z^{-1}$ and $\mathbf{R}(z) = \mathbf{R}_0 + \mathbf{R}_1 z$. The LP condition implies that [1]

$$\mathbf{E}_1 = \mathbf{D}_M\mathbf{E}_0\mathbf{J}_M \qquad \text{and} \qquad \mathbf{R}_1 = \mathbf{J}_M\mathbf{R}_0\mathbf{D}_M. \tag{29}$$

Equivalently, $\mathbf{E}_i$ and $\mathbf{R}_i$ (for $i = 0, 1$) should take the following forms

$$\mathbf{E}_0 = \begin{bmatrix} \mathbf{E}_u \\ \mathbf{E}_d \end{bmatrix}, \qquad \mathbf{E}_1 = \begin{bmatrix} \mathbf{E}_u\mathbf{J}_M \\ -\mathbf{E}_d\mathbf{J}_M \end{bmatrix}; \tag{30}$$

and

$$\mathbf{R}_0 = \begin{bmatrix} \mathbf{R}_l & \mathbf{R}_r \end{bmatrix}, \qquad \mathbf{R}_1 = \begin{bmatrix} \mathbf{J}_M\mathbf{R}_l & -\mathbf{J}_M\mathbf{R}_r \end{bmatrix}; \tag{31}$$

in which $\mathbf{E}_u$ and $\mathbf{E}_d$ are, respectively, the upper and lower submatrices of $\mathbf{E}_0$ with size of $M/2 \times M$ each, while $\mathbf{R}_l$ and $\mathbf{R}_r$ are, respectively, the left and right submatrices of $\mathbf{R}_0$ with size of $M \times M/2$ each. Besides, the PR property $\mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}_M$ leads to

$$\mathbf{E}(z)\mathbf{R}(z) = \mathbf{I}_M, \tag{32}$$

from which one can derive that

$$\mathbf{E}_0\mathbf{R}_0 + \mathbf{E}_1\mathbf{R}_1 = \mathbf{I}_M, \tag{33}$$

and

$$\mathbf{E}_0\mathbf{R}_1 = \mathbf{0}_M. \tag{34}$$

With (29), one can obtain

$$\mathbf{E}_0\mathbf{R}_1 = \mathbf{E}_0\mathbf{J}_M\mathbf{R}_0\mathbf{D}_M = \mathbf{0}_M \Longrightarrow \mathbf{E}_0\mathbf{J}_M\mathbf{R}_0 = \mathbf{0}_M. \tag{35}$$

Substituting (30) and (31) into (33) yields

$$2\mathbf{E}_u\mathbf{R}_l = 2\mathbf{E}_d\mathbf{R}_r = \mathbf{I}_{M/2}. \tag{36}$$

As for an $x \times y$ matrix $\mathbf{A}$ and a $y \times z$ matrix $\mathbf{B}$, $min\big(rank(\mathbf{A}), rank(\mathbf{B})\big) \geq \mathbf{rank}(\mathbf{AB})$, thus,

$$\begin{aligned} rank(\mathbf{E}_u) \geq M/2, \quad rank(\mathbf{E}_d) \geq M/2, \\ rank(\mathbf{R}_l) \geq M/2, \quad rank(\mathbf{R}_r) \geq M/2. \end{aligned} \tag{37}$$

Besides, for an $x \times y$ matrix $\mathbf{A}$, $rank(\mathbf{A}) \leq \mathbf{min(x, y)}$, hence,

$$\begin{aligned} rank(\mathbf{E}_u) \leq M/2, \quad rank(\mathbf{E}_d) \leq M/2, \\ rank(\mathbf{R}_l) \leq M/2, \quad rank(\mathbf{R}_r) \leq M/2. \end{aligned} \tag{38}$$

Based on (37) and (38), we can deduce that

$$\begin{aligned} &rank(\mathbf{E}_u) = rank(\mathbf{E}_d) \\ =&rank(\mathbf{R}_l) = rank(\mathbf{R}_d) = M/2, \end{aligned} \tag{39}$$

which means that all the $M/2$ rows of $\mathbf{E}_u$ and $\mathbf{E}_d$ must be independent, and all the $M/2$ columns of $\mathbf{R}_l$ and $\mathbf{R}_r$ are independent, too. Hence,

$$rank(\mathbf{E}_0) \geq M/2 \qquad \text{and} \qquad rank(\mathbf{R}_0) \geq M/2. \tag{40}$$

Furthermore, applying Sylvester's rank inequality [3] to (35) results in

$$\begin{aligned} & rank(\mathbf{E}_0) + rank(\mathbf{R}_0) - M \leq rank(\mathbf{E}_0\mathbf{J}_M\mathbf{R}_0) = 0 \\ \Longrightarrow \quad & rank(\mathbf{E}_0) + rank(\mathbf{R}_0) \leq M. \end{aligned} \tag{41}$$

From (40) and (41), we have

$$rank(\mathbf{E}_0) = rank(\mathbf{R}_0) = M/2. \tag{42}$$

Together, (39) and (42) mean that the row vectors of $\mathbf{E}_u$ and $\mathbf{E}_d$ must span the same space. Hence, $\mathbf{E}_u$ can be represented through a linear transform of $\mathbf{E}_d$. That is, there always exists an invertible matrix $\mathbf{U}_1$ such that (12), or equivalently, (10) exists, which further means that causality is met.

From the above discussions, Theorem II in [1] should be amended as follows.

Theorem 1: For any M-channel (M even) LPPRFB with all the analysis and synthesis filters of length $L = 2M$ each, the corresponding analysis polyphase matrix $\mathbf{E}(z)$ can always be factored as in (5).

IV. Conclusion

We have re-examined the completeness of the lattice factorization for M-channel (M even) LPPRFB with all the filters of the same length $L = KM$ in [1]. We point out that the original proof of completeness contains an incorrect assumption. Through counter examples, we show that the factorization is not complete for $K > 2$. Additionally, we prove that the factorization is complete when $K \leq 2$. The complete factorization of the general $L = KM$ ($K > 2$) case is still an open problem.

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