Department of Electrical and Computer Engineering
The Johns Hopkins University
520.137 Introduction to Electrical and Computer Engineering – Fall 2018

Homework Assignment III
Solutions

1. Consider a simple RC circuit with a DC voltage source V_S sketched below. At some time in the distant past, the DC voltage source was connected to the circuit to charge up the capacitor C. You can assume that C has been fully charged prior to $t = 0$.

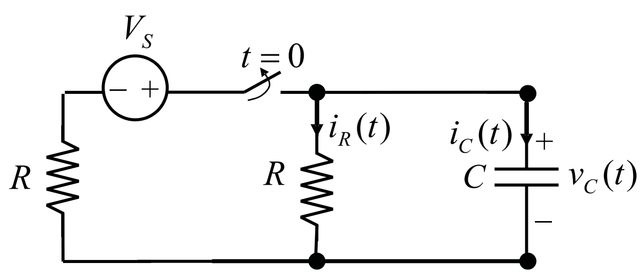

Figure 1: RC Circuit with a DC Voltage Source for Problem 1.

   (a) Before $t = 0$, the switch is closed. Assume that the capacitor has been fully charged prior to $t = 0$. Then it behaves like an open circuit. This is shown in figure 2. This means for $t \leq 0$,

$$V_C(t) = i_R(t) \cdot R = \frac{V_S}{R+R} \times R = \frac{V_S}{2}$$

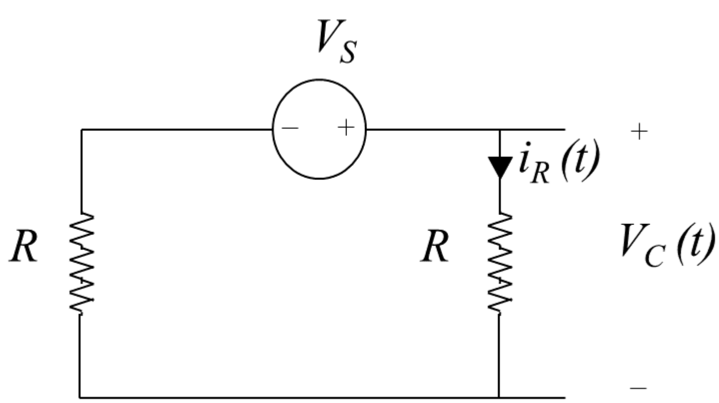

Figure 2: Circuit for Problem 1 before $t = 0$ with fully charged capacitor

(b) After the switch opens at $t = 0$, the circuit is shown in figure 3. Using KCL, KVL, and the capacitor equations for the simple circuit we get,

$$
\begin{aligned}
i_R(t) + i_C(t) &= 0 \\
V_C(t) - i_R(t)R &= 0 \\
i_C(t) &= C\frac{dV_C(t)}{dt} \\
\implies V_C(t) + Ri_C(t) &= 0 \\
\implies RC\frac{dV_C(t)}{dt} + V_C(t) &= 0
\end{aligned}
$$

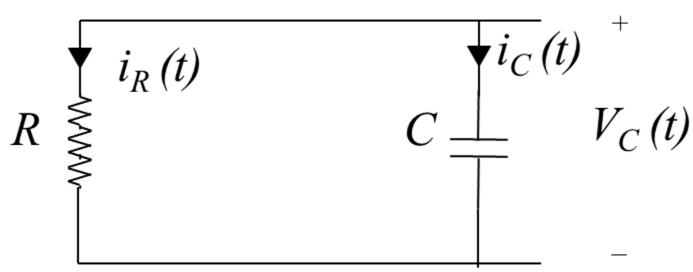

Figure 3: Circuit for Problem 1 after $t = 0$

(c) For $t \geq 0$, We can solve the differential equation derived in part (b) to get $V_C(t) = \frac{V_S}{2}e^{-\frac{t}{RC}}$. The graph is sketched in figure 4

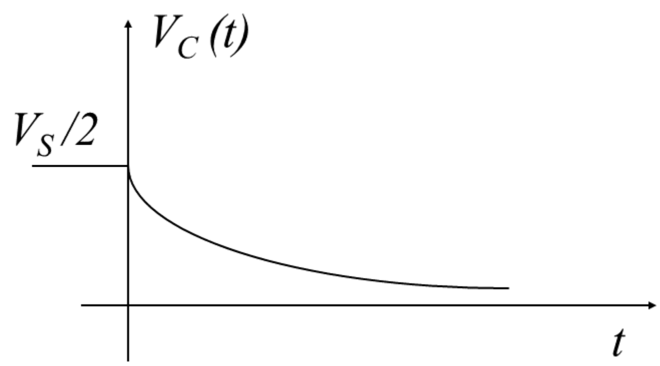

Figure 4: Problem 1 - Plot of $V_C(t)$

(d) For finding the current i_R, we can use the current law to get:

$$\begin{aligned} i_R(t) &= -i_C(t) \\ &= -C\frac{dV_C(t)}{dt} \\ &= -C \times \frac{V_S}{2} \times \frac{-1}{RC} e^{-\frac{t}{RC}} \\ &= \frac{V_S}{2R} e^{-\frac{t}{RC}} \end{aligned}$$

The graph is sketched in figure 5

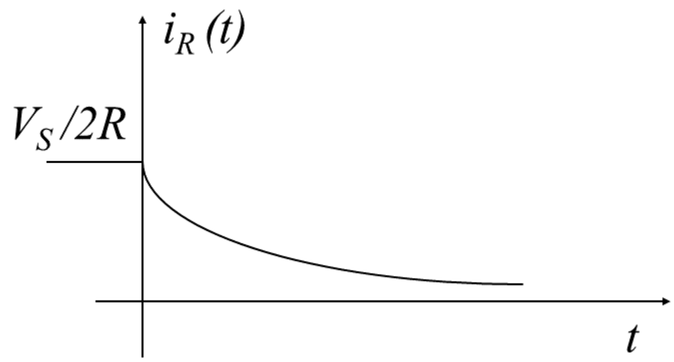

Figure 5: Problem 1 - Plot of $i_R(t)$

(e) At $t = T >> RC$, the switch is closed. The circuit is drawn in figure 6. Using Kirchoff's Voltage law at loop L_1, we get:

$$\begin{aligned} i_R(t)R + (i_C(t) + i_R(t))R = V_S \\ \implies i_R(t) = \frac{1}{2}\left(\frac{V_S}{R} - i_C(t)\right) \end{aligned}$$

Using KVL at L_2:

$$\begin{aligned} i_R(t)R = V_C(t) \\ \implies \frac{1}{2}\left(\frac{V_S}{R} - i_C(t)\right) R = V_C(t) \end{aligned}$$

Using the capacitor relationship between i_C and V_C, we can get:

$$\begin{aligned} \frac{V_S}{2} = \frac{RC}{2}\frac{dV_C(t)}{dt} + V_C(t) \\ \implies V_C(t) = \frac{V_S}{2}\left(1 - e^{-\frac{2t}{RC}}\right) \end{aligned}$$

The graph of V_C is sketched in figure 7

We could also solve this problem using the Thevenin equivalent circuit looking into the terminals of the capacitor. The Thevenin equivalent circuit looking from A,B as marked in figure 8 has:

$$V_{Th} = V_{AB} = \frac{V_S}{2R} \times R = \frac{V_S}{2}$$

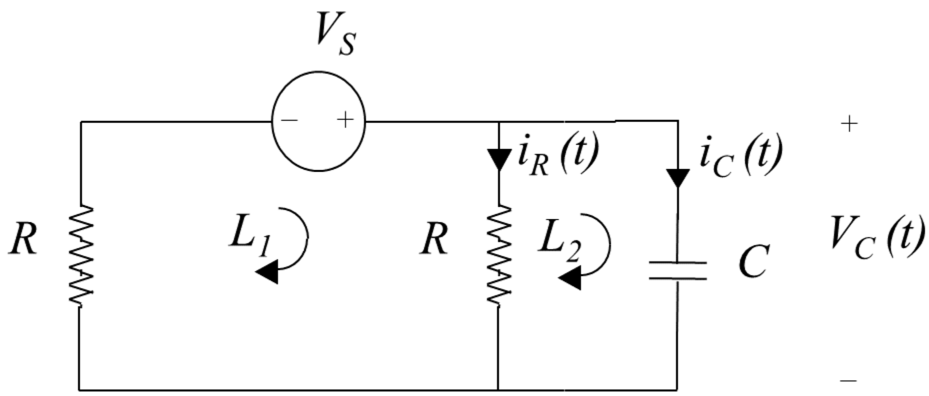

Figure 6: Circuit for Problem 1 when switch is closed at $t = T >> RC$

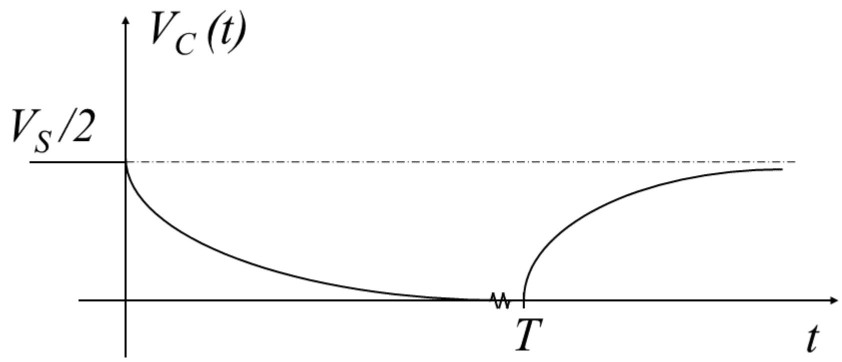

Figure 7: Problem 1 - Plot of $V_C(t)$ for part (e)

$$R_{Th} = R||R = \frac{R}{2}$$

From Kirchoff's Voltage law, and the capacitor equations in the Thevenin equivalent circuit (figure 9), we get:

$$V_{Th} = R_{Th}C\frac{dV_C(t)}{dt} + V_C(t)$$
$$\implies \frac{V_S}{2} = \frac{RC}{2}\frac{dV_C(t)}{dt} + V_C(t)$$

Which is the same differential equation as the one we obtained earlier.

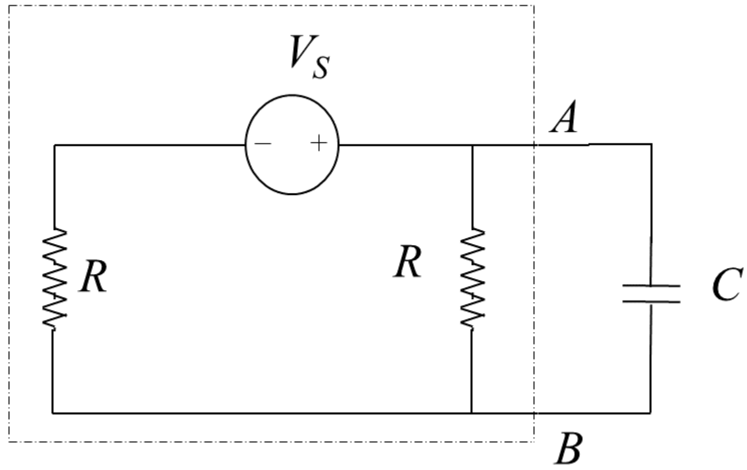

Figure 8: Circuit for Problem 1 looking into terminals of capacitor (A and B)

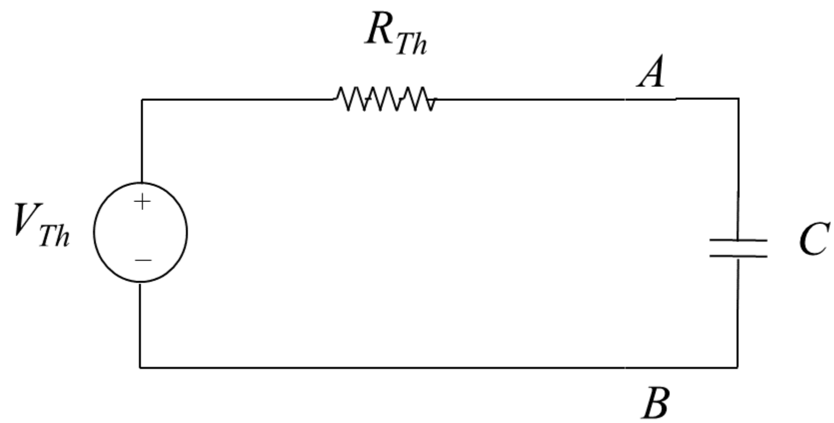

Figure 9: Thevenin Equivalent Circuit for Problem 1

2. Consider an RC circuit with a DC current source I_S sketched below in figure 10. At some time in the distant past, the DC current source was connected to the circuit to charge up the capacitor C (the switch stays open). You can assume that C has been fully charged prior to $t = 0$.
   (a) After the switch closes at $t > 0$, the circuit is as depicted in figure 11.
   Using KVL in loop L, and using the fact that $i_R = i_C$:

$$2Ri_R(t) + V_C(t) = 0$$
$$\implies 2RC\frac{dV_C(t)}{dt} + V_C(t) = 0$$

   (b) We are given that $R = 5K\Omega$, $C = 100\mu F$, and $I_S = 2mA$. Before $t = 0$, the switch is open. Assuming that the capacitor is fully charged, the circuit is depicted in figure 12. The voltage drop across the capacitor is the same as that across the resistor R. For $t \leq 0$,

$$V_C(t) = V_R(t) = I_S \times R = 10V$$

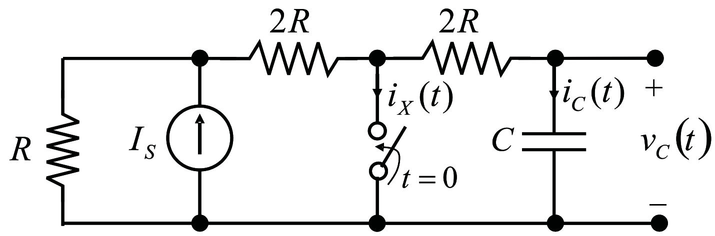

Figure 10: RC Circuit for Problem 2 with a DC Current Source for Problem 2.

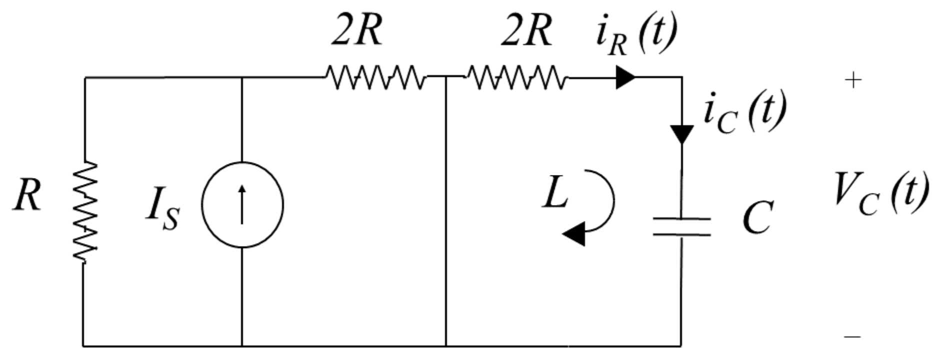

Figure 11: Circuit for Problem 2 when switch is closed at $t = 0$

(c) From the differential equation in Part (a) and the initial condition in Part (b), we can get $V_C(t) = 10e^{-\frac{t}{2RC}}V = 10e^{-t}V$. The voltage is sketched in figure 13

(d) Using the capacitor equation, for $t \geq 0$,

$$i_C(t) = C\frac{dV_C(t)}{dt} = -10Ce^{-t}A$$

$$\implies i_C(t) = -e^{-t}mA$$

The current is sketched in figure 14

(e) A long time after the switch closes, the circuit is as depicted in figure 15. This is a current divider circuit between R and $2R$.
We get $i_X(t) = \frac{R}{R+2R}I_S = \frac{1}{3}I_S = \frac{2}{3}mA$

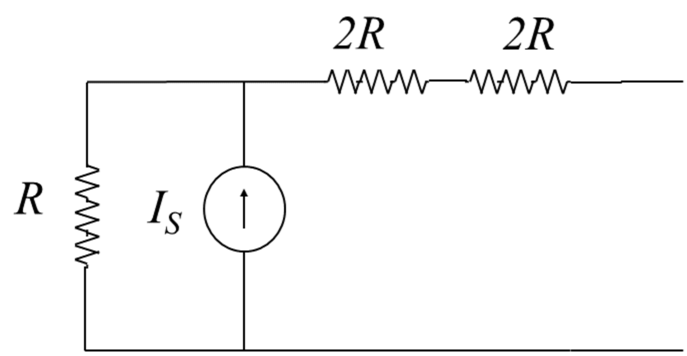

Figure 12: Circuit for Problem 2 when switch is open during $t \leq 0$

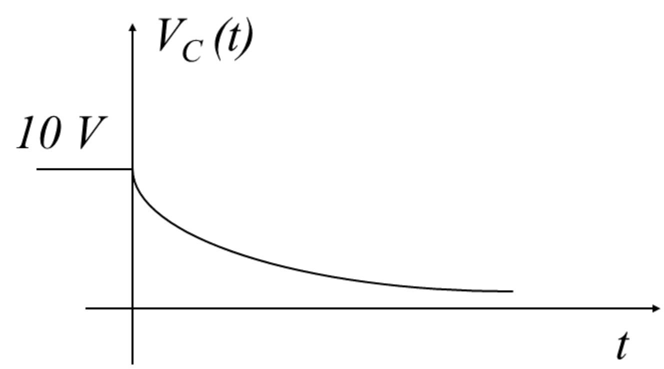

Figure 13: Problem 2 - Plot for $V_C(t)$

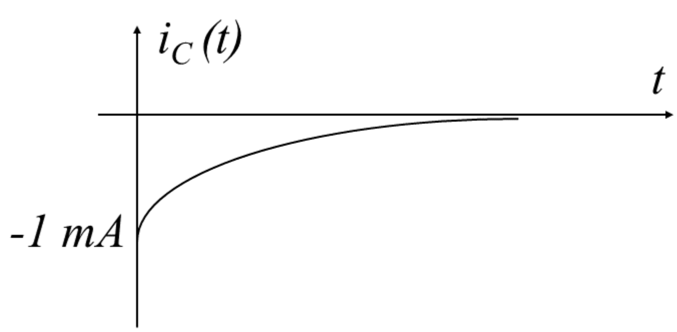

Figure 14: Problem 2 - Plot for $i_C(t)$

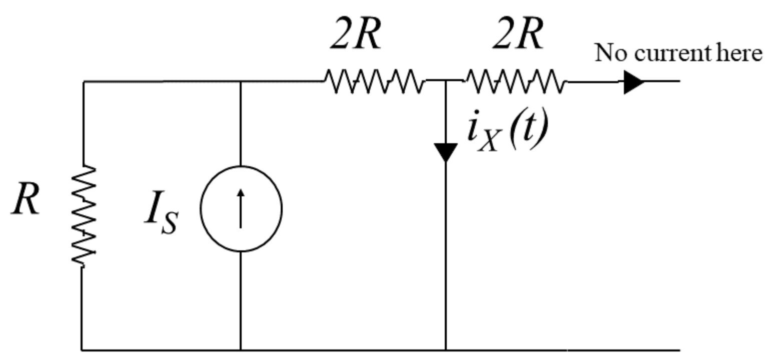

Figure 15: Circuit for Problem 2 a long time after the switch is closed