Department of Electrical and Computer Engineering
The Johns Hopkins University
520.646 Wavelets and Filter Banks – Fall 2018

Problem Set VIII

1. Consider the analysis polyphase structure given in Figure 1 below where θ_0 and θ_1 are rotation angles, i.e.,

$$\mathbf{\Theta}_i = \begin{bmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{bmatrix} = \begin{bmatrix} c_i & s_i \\ -s_i & c_i \end{bmatrix}.$$

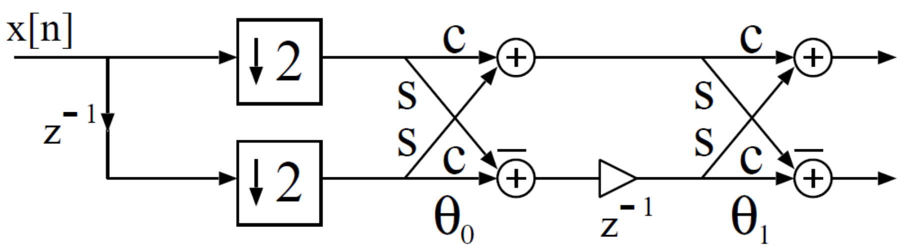

Figure 1: A 2-channel analysis filter bank in polyphase form.

   (a) Is perfect reconstruction possible, i.e., $\hat{x}[n] = x[n-\ell]$? Design the *causal* synthesis polyphase matrix. Is there any restriction on the values of θ_0 and θ_1?

   (b) Draw the causal reconstruction (synthesis) bank in polyphase form.

   (c) Find the corresponding analysis and synthesis filters. Show that they form an orthogonal filter bank.

   (d) Prove that the sum of the rotation angles is $\frac{\pi}{4}$ in order for $H_0(e^{j\omega})$ to have one zero at $\omega = \pi$.

   (e) What about two zeros at $\omega = \pi$.?

2. You are given a 2-channel perfect reconstruction filter bank ($\hat{x}[n] = x[n-\ell]$) with filters $H_0(z), H_1(z), F_0(z), F_1(z)$ and associated polyphase matrices $\mathbf{H}_p(z)$ and $\mathbf{F}_p(z)$. Now, consider the addition of a lifting step

$$\begin{bmatrix} 1 & 0 \\ -P(z) & 1 \end{bmatrix}$$

as shown in Figure 2 below.

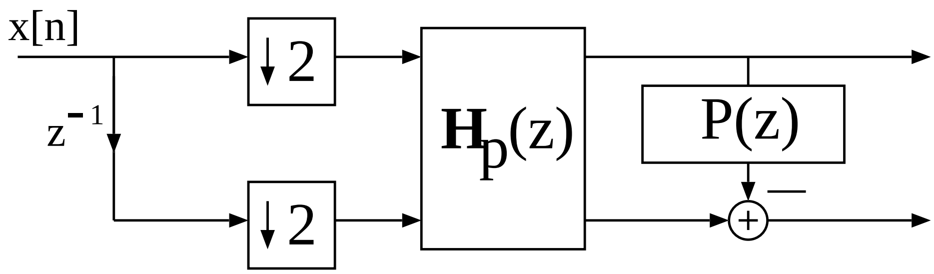

Figure 2: A construction using lifting.

(a) Draw the corresponding new synthesis bank and find the new polyphase matrices $\mathbf{H}'_p(z)$ and $\mathbf{F}'_p(z)$.

(b) Show that the new analysis filters are

$$H'_0(z) = H_0(z) \quad \text{and} \quad H'_1(z) = H_1(z) - P(z^2)H_0(z).$$

(c) Find the new synthesis filters. Find the time-domain relationships between new four filters and the given filters.

(d) Show that the addition of the lifting step leaves the analysis scaling function $\phi(t)$ intact but yields a new analysis wavelet function

$$\psi'(t) \;=\; \psi(t) - \sqrt{2}\sum_k \; p[k] \; \phi(t-k).$$

(e) How does the addition of the lifting step alter the degree of regularity in the analysis and the synthesis bank?

3. Consider the lifting structure illustrated in Figure 3 below where $\{\alpha, \beta, \gamma, \delta, \zeta\}$ are the free parameters.

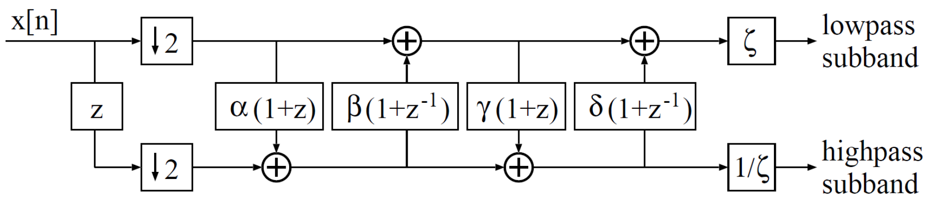

Figure 3: A lifting structure with 4 lifting steps.

(a) Prove that the structure in Figure 3 always generates symmetric filters regardless of the parameter set $\{\alpha, \beta, \gamma, \delta, \zeta\}$.

(b) Find the synthesis polyphase matrix and draw the appropriate reconstructrion stage such that $\hat{x}[n] = x[n]$. Find the condition(s) on the parameter set $\{\alpha, \beta, \gamma, \delta, \zeta\}$ such that perfect reconstruction is always guaranteed.

(c) Prove that if we desire the same number of vanishing moments on both bank, then the maximum achievable number in this case is four. *Hint: think of $P_0(z)$!*

(d) What are the average number of additions and multiplications do we need to compute one wavelet coefficient (subband sample) using the lifting structure in a 1-level 1D decomposition? Compare these with the numbers obtained from direct convolution implementation.

Due date: **Monday Nov. 26** in class